

KNAPSACK PROBLEMS IN PRODUCTS OF GROUPS

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ABSTRACT. The classic knapsack and related problems have natural generalizations to arbitrary (non-commutative) groups, collectively called knapsack-type problems in groups. We study the effect of free and direct products on their time complexity. We show that free products in certain sense preserve time complexity of knapsack-type problems, while direct products may amplify it. Our methods allow to obtain complexity results for rational subset membership problem in amalgamated free products over finite subgroups.

Keywords. Subset sum problem, knapsack problem, bounded subgroup membership problem, rational subset membership problem, free products, direct products, p hyperbolic groups, p nilpotent groups.

2010 Mathematics Subject Classification. 03D15, 20F65, 20F10, 68Q45.

1. INTRODUCTION

In [14], the authors introduce a number of certain decision, search and optimization algorithmic problems in groups, such as the subset sum problem, the knapsack problem, and the bounded submonoid membership problem (see Section 1.1 for definitions). These problems are collectively referred to as *knapsack-type* problems and deal with different generalizations of the classic knapsack and subset sum problems over \mathbb{Z} to the case of arbitrary groups. In the same work, the authors study time complexity of such problems for various classes of groups, for example for nilpotent, metabelian, hyperbolic groups. With that collection of results in mind, it is natural to ask what is the effect of group constructions on the complexity of knapsack-type problems, primarily the subset sum problem. In the present paper we address this question in its basic variation, for the case of free and direct products of groups.

Solutions to many algorithmic problems carry over from groups to their free products without much difficulty. It certainly is the case with classic decision problems in groups such as the word, conjugacy [8, for instance] and membership [12] problems. In some sense, the same expectations are satisfied with knapsack-type problems, albeit not in an entirely straightforward fashion. It turns out that knapsack-type problems such as the aforementioned subset sum problem, the bounded knapsack problem, and the bounded submonoid membership problem share a certain common ground that allows to approach these problems in a unified fashion, and to carry solutions of these problems over to free products. Thus, our research both presents certain known facts about these algorithmic problems in a new light, and widens the class of groups with known complexity of the knapsack-type problems. Our methods apply more generally, which allows us to establish in Section 4 complexity results for certain decision problems, including the rational subset membership problem, in free products of groups with finite amalgamated subgroups.

The work of the third author was partially supported by NSF grant DMS-1318716.

Algorithmic problems in a direct product of groups can be dramatically more complex than in either factor, as is the case with the membership problem, first shown in [11]. By contrast, the word and conjugacy problems in direct products easily reduce to those in the factors. In Section 3 we show that direct product does not preserve polynomial time subset sum problem (unless $\mathbf{P} = \mathbf{NP}$). Thus, the subset sum problem occupies an interesting position, exhibiting features of both word problem and membership problem; on the one hand, its decidability clearly carries immediately from factors to the direct product, while, on the other hand, its time complexity can increase dramatically.

Below we provide basic definitions and some of the immediate properties of the problems mentioned above.

1.1. Preliminaries. In this paper we follow terminology and notation introduced in [14]. For convenience, below we formulate the algorithmic problems mentioned in Section 1. We collectively refer to these problems as *knapsack-type* problems in groups.

Elements in a group G generated by a finite or countable set X are given as words over the alphabet $X \cup X^{-1}$. As we explain in the end of this section, the choice of a finite X does not affect complexity of the problems we formulate below. Therefore, we omit the generating set from notation.

Consider the following decision problem. Given $g_1, \dots, g_k, g \in G$, and m that is either a unary positive integer or the symbol ∞ , decide if

$$(1) \quad g = g_1^{\varepsilon_1} \dots g_k^{\varepsilon_k}$$

for some integers $\varepsilon_1, \dots, \varepsilon_k$ such that $0 \leq \varepsilon_j \leq m$ for all $j = 1, 2, \dots, k$. Depending on m , special cases of this problem are called:

($m = \infty$): The knapsack problem $\mathbf{KP}(G)$. We omit ∞ from notation, so the input of $\mathbf{KP}(G)$ is a tuple $g_1, \dots, g_k, g \in G$.

($m < \infty$): The bounded knapsack problem $\mathbf{BKP}(G)$. The input of $\mathbf{BKP}(G)$ is a tuple $g_1, \dots, g_k, g \in G$, and a number 1^m .

($m = 1$): The subset sum problem $\mathbf{SSP}(G)$. We omit $m = 1$ from notation, so the input of $\mathbf{SSP}(G)$ is a tuple $g_1, \dots, g_k, g \in G$.

One may note that $\mathbf{BKP}(G)$ is \mathbf{P} -time equivalent to $\mathbf{SSP}(G)$ (we recall the definition of \mathbf{P} -time reduction below), so it suffices for our purposes to consider only \mathbf{SSP} in groups.

The submonoid membership problem formulated below is equivalent to the knapsack problem in the classic (abelian) case, but in general it is a completely different problem that is of prime interest in algebra.

Consider the following decision problem. Given $g_1, \dots, g_k, g \in G$, and m that is either a unary positive integer or the symbol ∞ , decide if the following equality holds for some $g_{i_1}, \dots, g_{i_s} \in \{g_1, \dots, g_k\}$ and some integer s such that $0 \leq s \leq m$:

$$(2) \quad g = g_{i_1} \dots g_{i_s}.$$

Depending on m , special cases of this problem are called:

($m = \infty$): Submonoid membership problem $\mathbf{SMP}(G)$. We omit ∞ from notation, so the input of $\mathbf{SMP}(G)$ is a tuple $g_1, \dots, g_k, g \in G$.

($m < \infty$): Bounded submonoid membership problem $\mathbf{BSMP}(G)$. The input of this problem is a tuple $g_1, \dots, g_k, g \in G$ and a number 1^m .

The restriction of **SMP** to the case when the set of generators $\{g_1, \dots, g_n\}$ is closed under inversion (so the submonoid is actually a subgroup of G) is a well-known problem in group theory, the *uniform subgroup membership problem* in G (see e.g. [10]).

In general, both **SMP** and **KP** can be undecidable in a group with decidable word problem, for instance, a group with decidable word problem, but undecidable membership in cyclic subgroups is constructed in [15]. Groups with undecidable **SMP** or **KP** are not necessarily quite so hand-crafted; for example, famous Mikhailova construction [12] shows that subgroup membership and, therefore, submonoid membership is undecidable in a direct product of a free group with itself, and recent works show that the knapsack problem is undecidable in integer unitriangular groups of sufficiently large size [7] and, more broadly, large family of nilpotent groups of class at least 2 [13]. *Bounded* versions of **KP** and **SMP** are at least always decidable in groups where the word problem is.

Recall that a decision problem can be denoted as (I, D) , where I is the set of all instances of the problem, and $D \subseteq I$ is the set of positive ones. A decision problem (I_1, D_1) is **P-time reducible** to a problem (I_2, D_2) if there is a **P-time** computable function $f : I_1 \rightarrow I_2$ such that for any $u \in I_1$ one has $u \in D_1 \iff f(u) \in D_2$. Such reductions are usually called either *many-to-one P-time reductions* or *Karp reductions*. Since we mainly use this type of reduction we omit “many-to-one” from the name and call them **P-time reductions**. We say that two problems are **P-time equivalent** if each of them **P-time** reduces to the other. Aside from many-to-one reductions, we use the so-called Cook reductions. That is, we say that a decision problem (I_1, D_1) is **P-time Cook reducible** to a problem (I_2, D_2) if there is an algorithm that solves problem (I_1, D_1) using a polynomial number of calls to a subroutine for problem (I_2, D_2) , and polynomial time outside of those subroutine calls. Correspondingly, we say that two problems are **P-time Cook equivalent** if each of them **P-time Cook** reduces to the other. In our present work we are primarily interested in establishing whether certain problems are **P-time** decidable or **NP-complete**, and thus we make no special effort to use one of the two types of reduction over the other.

Finally we mention that for $\Pi \in \{\mathbf{SSP}, \mathbf{KP}, \mathbf{BKP}, \mathbf{SMP}, \mathbf{BSMP}\}$, the problem $\Pi(G, X_1)$ is **P-time** (in fact, linear time) equivalent to $\Pi(G, X_2)$ for any finite generating sets X_1, X_2 , i.e. the time complexity of $\Pi(G)$ does not depend on the choice of a finite generating set of a given group G . For this reason, we simply write $\Pi(G)$ instead of $\Pi(G, X)$, implying an arbitrary finite generating set. We refer the reader to [14] for the proof of the above statement, further information concerning knapsack-type problems and their variations in groups, details regarding their algorithmic set-up, and the corresponding basic facts. Here we limit ourselves to mentioning that all of the above problems can be regarded as special cases of the uniform rational subset membership problem for G , with input given by a (non-deterministic) finite state automation and a group element. We take advantage of this viewpoint in Sections 2 and 4.

1.2. Results and open questions. Primary goal of the present work is to answer the following basic questions about complexity of the knapsack problem (**KP**) and the subset sum problem (**SSP**) in free and direct products.

- Q1. Assuming $\mathbf{SSP}(G), \mathbf{SSP}(H) \in \mathbf{P}$, is it true that $\mathbf{SSP}(G * H) \in \mathbf{P}$?
- Q2. Assuming $\mathbf{KP}(G), \mathbf{KP}(H) \in \mathbf{P}$, is it true that $\mathbf{KP}(G * H) \in \mathbf{P}$?

Q3. Assuming $\mathbf{SSP}(G), \mathbf{SSP}(H) \in \mathbf{P}$, is it true that $\mathbf{SSP}(G \times H) \in \mathbf{P}$?

Q4. Assuming $\mathbf{KP}(G), \mathbf{KP}(H) \in \mathbf{P}$, is it true that $\mathbf{KP}(G \times H) \in \mathbf{P}$?

We give a partial positive answer to the question Q1. To elaborate, in the context of studying properties of the subset sum problem in free products, it is natural to view this problem, along with the bounded knapsack and the bounded submonoid membership problems, as special cases of a more general algorithmic problem formulated in terms of graphs labeled by group elements, which we call the *acyclic graph word problem* (**AGWP**), introduced in Section 2 below.

In the same section we also establish connection between **SSP**, **BKP**, **BSMP** and **AGWP**, as shown in Figure 1, and show that **AGWP** is \mathbf{P} -time solvable in all

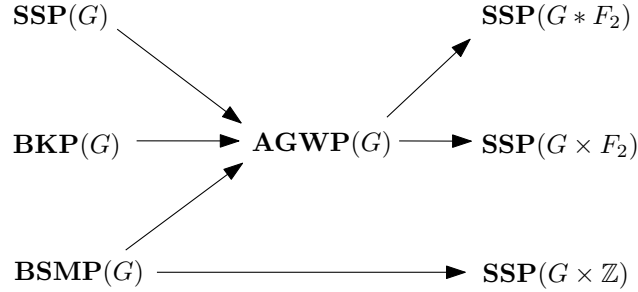


FIGURE 1. Connection between algorithmic problems. Arrows denote \mathbf{P} -time (Karp or Cook) reductions.

known groups where **SSP** is. Further, we show in Section 4 that **AGWP** in free products with amalgamation over a finite subgroup is \mathbf{P} -time Cook reducible to **AGWP** in the factors. As a consequence, we obtain that **SSP**, **BKP**, **BSMP** are polynomial time solvable in a wide class of groups. Tools used in Section 4 allow to produce similar complexity estimates for a wider class of problems, including rational subset membership problem.

As an answer to Q2, we show in Section 5 that the (unbounded) knapsack problem in a free product has a polynomial bound on length of solution if and only if $\mathbf{KP}(G)$ and $\mathbf{KP}(H)$ have such a bound. This condition allows a \mathbf{P} -time reduction of $\mathbf{KP}(G * H)$ to $\mathbf{AGWP}(G * H)$, and, therefore, a \mathbf{P} -time solution of $\mathbf{KP}(G * H)$ when $\mathbf{AGWP}(G), \mathbf{AGWP}(H) \in \mathbf{P}$.

We observe that the same is false for direct products, i.e. that $\mathbf{AGWP}(G) \in \mathbf{P}$, $\mathbf{AGWP}(H) \in \mathbf{P}$ does not imply $\mathbf{AGWP}(G \times H) \in \mathbf{P}$ (unless $\mathbf{P} = \mathbf{NP}$). In Section 3 we prove a similar result for $\mathbf{SSP}(G \times H)$, which negatively resolves question Q3.

The following questions remain open.

- OQ1. Is there a group G with $\mathbf{SSP}(G) \in \mathbf{P}$ and \mathbf{NP} -hard $\mathbf{AGWP}(G)$ (cf. results shown in Figure 1)?
- OQ2. Let F_2 be a free group of rank 2. Is $\mathbf{SSP}(F_2 \times F_2)$ \mathbf{NP} -complete (cf. Proposition 8)?

2. ACYCLIC GRAPH WORD PROBLEM

Let G be a group and $X = \{x_1, \dots, x_n\}$ a generating set for G .

The acyclic graph word problem $\mathbf{AGWP}(G, X)$: Given an acyclic directed graph Γ labeled by letters in $X \cup X^{-1} \cup \{\varepsilon\}$ with two marked vertices, α and ω , decide whether there is an oriented path in Γ from α to ω labeled by a word w such that $w = 1$ in G .

An immediate observation is that this problem, like all the problems introduced in Section 1.1, is a special case of the uniform membership problem in a rational subset of G . We elaborate on that in Section 4.

Let graph Γ have n vertices and m edges. Define $\text{size}(\Gamma)$ to be $m + n$. Let total word length of labels of edges of Γ be l . For a given instance of $\mathbf{AGWP}(G, X)$, its *size* is the value $m + n + l$. With a slight abuse of terminology, we will also sometimes use labels that are words rather than letters in $X \cup X^{-1} \cup \{\varepsilon\}$. Note that given such a graph Γ of size $m + n + l$, subdividing its edges we can obtain a graph Γ' where edges are labeled by letters, with $m + n + l \leq \text{size}(\Gamma') \leq (m + l) + (n + l) + l$. Therefore, $1/3 \text{size}(\Gamma') \leq \text{size}(\Gamma) \leq \text{size}(\Gamma')$, so such an abuse of terminology results in distorting the size of an instance of $\mathbf{AGWP}(G, X)$ by a factor of at most 3.

Note that by a standard argument, if X_1 and X_2 are two finite generating sets for a group G , the problems $\mathbf{AGWP}(G, X_1)$ and $\mathbf{AGWP}(G, X_2)$ are **P**-time (in fact, linear time) equivalent. In this sense, the complexity of \mathbf{AGWP} in a group G does not depend on the choice of a finite generating set. In the sequel we write $\mathbf{AGWP}(G)$ instead of $\mathbf{AGWP}(G, X)$, implying an arbitrary finite generating set. As shown in [14], complexity of algorithmic problems in a group can change dramatically depending on a choice of infinite generating set. We do not consider infinite generating sets in the present work, and refer the reader to the above paper for details of treating those.

The above definition of \mathbf{AGWP} can be given for an arbitrary finitely generated monoid G , rather than a group. The input in such case is a pair (Γ, w_0) consisting of a directed graph Γ labeled by letters in $X \cup \{\varepsilon\}$, and a word w_0 in X . The problem asks to decide if a word w such that $w = w_0$ in G is readable as a label of a path in Γ from α to ω . Given the immediate linear equivalence of the two formulations in the case of a group, we use the same notation $\mathbf{AGWP}(G)$ for this problem. Note that the problems introduced in Section 1.1 (subset sum problem, knapsack problem, etc) also can be formulated for a monoid.

We make a note of the following obvious property of \mathbf{AGWP} .

Proposition 1. Let G be a finitely generated monoid and $H \leq G$ its finitely generated submonoid. Then $\mathbf{AGWP}(H)$ is **P**-time reducible to $\mathbf{AGWP}(G)$. In particular,

- (1) If $\mathbf{AGWP}(H)$ is **NP**-hard then $\mathbf{AGWP}(G)$ is **NP**-hard.
- (2) If $\mathbf{AGWP}(G) \in \mathbf{P}$ then $\mathbf{AGWP}(H) \in \mathbf{P}$.

Methods used in [14] to treat **SSP** and **BSMP** in hyperbolic and nilpotent groups can be easily adjusted to treat \mathbf{AGWP} as well, as we see in the two following statements.

Proposition 2. $\mathbf{AGWP}(G) \in \mathbf{P}$ for every finitely generated virtually nilpotent group G .

Proof. The statement follows from Theorem 2.1 below. □

Proposition 3. $\mathbf{AGWP}(G) \in \mathbf{P}$ for every finitely generated hyperbolic group G .

Proof. The statement follows immediately from [14, Proposition 5.5]. \square

Here we would also like to note that since the word problem straightforwardly reduces to **AGWP**, an obvious prerequisite for **AGWP**(G) to belong to **NP** is to have a polynomial time word problem. Therefore, in the context of investigating time complexity of **AGWP** we are primarily interested in such groups.

We also note that adding or eliminating a direct factor \mathbb{Z} or, more generally, a finitely generated virtually nilpotent group, does not change complexity of **AGWP**.

Theorem 2.1. *Let G be a finitely generated monoid and N a finitely generated virtually nilpotent group. Then **AGWP**(G) and **AGWP**($G \times N$) are **P**-time equivalent.*

Proof. We only have to show the reduction of **AGWP**($G \times N$) to **AGWP**(G), since the other direction is immediate by Proposition 1.

Since the complexity of **AGWP** in a given monoid does not depend on a finite generating set, we assume that the group $G \times N$ is generated by finitely many elements $(g_1, 1_N), (g_2, 1_N), \dots$, and $(1_G, h_1), (1_G, h_2), \dots$. Consider an arbitrary instance $(\Gamma, \alpha, \omega, (\hat{g}, \hat{h}))$ of **AGWP**($G \times N$), where $\Gamma = (V, E)$. Consider a graph $\Gamma^* = (V^*, E^*)$, where $V^* = V \times N$ and

$$E^* = \left\{ (v, h) \xrightarrow{g} (v', hh') \mid \text{for every } (v, h) \in V^* \text{ and } v \xrightarrow{(g, h')} v' \in E \right\}.$$

where (g, h') denotes an element of $G \times N$, with $g \in G$, $h' \in \{1_N, h_i, h_i^{-1}\}$. Let $\Gamma' = (V', E')$ be the connected component of Γ^* containing $(\alpha, 1_N) \in V^*$ (or the subgraph of Γ^* induced by all vertices in Γ^* that can be reached from $(\alpha, 1_N)$). It is easy to see that

$$|V'| \leq |V| \cdot B_{|E|},$$

where $B_{|E|}$ is the ball of radius $|E|$ in the Cayley graph of N relative to generators $\{h_i\}$. Since the group N has polynomial growth [16] and polynomial time decidable word problem (in fact, real time by [3]), the graph Γ' can be constructed in a straightforward way in polynomial time. Finally, it follows from the construction of Γ' that $(\Gamma, \alpha, \omega, (\hat{g}, \hat{h}))$ is a positive instance of **AGWP**($G \times N$) if and only if $(\Gamma', (\alpha, 1_N), (\omega, \hat{h}), \hat{g})$ is a positive instance of **AGWP**(G). \square

The above statement and Proposition 3 provide the following corollary.

Corollary 2.2. **AGWP**($F_2 \times N$) \in **P** for every finitely generated virtually nilpotent group N .

Here, and everywhere below, F_2 denotes the free group of rank 2.

2.1. Connection between acyclic graph word problem and knapsack-type problems. The subset sum problem (**SSP**), the bounded knapsack problem (**BKP**), and the bounded submonoid membership problem (**BSMP**) in a monoid G reduce easily to the acyclic graph word problem (**AGWP**).

Proposition 4. Let G be a finitely generated monoid. **SSP**(G), **BKP**(G), **BSMP**(G) are **P**-time reducible to **AGWP**(G).

Proof. The proof below is for the case of a group G . The case of a monoid G is treated in the same way with obvious adjustments.

Let w_1, w_2, \dots, w_k, w be an input of $\mathbf{SSP}(G)$. Consider the graph $\Gamma = \Gamma(w_1, \dots, w_k, w)$ shown in the Figure 2. We see immediately that (w_1, \dots, w_k, w) is a positive instance of $\mathbf{SSP}(G)$ if and only if Γ is a positive instance of $\mathbf{AGWP}(G)$. Since



FIGURE 2. Graph $\Gamma(w_1, w_2, \dots, w_k, w)$, Proposition 4.

$\mathbf{BKP}(G)$ \mathbf{P} -time reduces to $\mathbf{SSP}(G)$ (see [14]), it is only left to prove that $\mathbf{BSMP}(G)$ reduces to $\mathbf{AGWP}(G)$. Indeed, let $(w_1, w_2, \dots, w_k, w, 1^n)$ be an input of $\mathbf{BSMP}(G)$. Consider the graph $\Delta = \Delta(w_1, w_2, \dots, w_k, w, 1^n)$ shown in Figure 3. It is easy to



FIGURE 3. Graph $\Delta(w_1, w_2, \dots, w_k, w, 1^n)$, Proposition 4. There are $n + 2$ vertices in the graph.

see that $(w_1, w_2, \dots, w_k, w, 1^n)$ is a positive instance of $\mathbf{BSMP}(G)$ if and only if Δ is a positive instance of \mathbf{AGWP} . \square

Given Proposition 4 and results of [14], we immediately see that \mathbf{AGWP} is \mathbf{NP} -complete in the following cases;

- certain metabelian groups (finitely generated free metabelian groups, wreath products of two infinite abelian groups, Baumslag's group $B = \langle a, s, t \mid [a, a^t] = 1, [s, t] = 1, a^s = aa^t \rangle$, Baumslag–Solitar groups $BS(m, n)$ with $|m| \neq |n|$, $m, n \neq 0$),
- Thompson's group F ,
- $F_2 \times F_2$,
- linear groups $GL(n, \mathbb{Z})$ with $n \geq 4$,
- braid groups B_n with $n \geq 5$ (by [9]),
- graph groups whose graph contains an induced square C_4 .

While it still remains to be seen whether \mathbf{AGWP} reduces to either of the problems in Proposition 4, we make note of the following two observations. First, in every case when it is known that those problems are \mathbf{P} -time, so is \mathbf{AGWP} , as shown in Propositions 2 and 3. The second observation is that for a given group G , $\mathbf{AGWP}(G)$ \mathbf{P} -time reduces to either of the problems $\mathbf{SSP}(G * F_2)$ or $\mathbf{SSP}(G \times F_2)$.

Proposition 5. Let G be a finitely generated monoid.

- (1) $\mathbf{AGWP}(G)$ is \mathbf{P} -time reducible to $\mathbf{SSP}(G * F_2)$.
- (2) $\mathbf{AGWP}(G)$ is \mathbf{P} -time reducible to $\mathbf{SSP}(G \times F_2)$.

Proof. Let Γ be a given directed acyclic graph on n vertices with edges labeled by group words in a generating set X of the group G . We start by organizing a topological sorting on Γ , that is enumerating vertices of Γ by symbols V_1 through

V_n so that if there is a path in Γ from V_i to V_j then $i \leq j$. This can be done in a time linear in $\text{size}(\Gamma)$ by [4]. We assume $\alpha = V_1$ and $\omega = V_n$, otherwise discarding unnecessary vertices. We perform a similar ordering of edges, i.e., we enumerate them by symbols E_1, \dots, E_m so that if there is a path in Γ whose first edge is E_i and the last edge E_j , then $i \leq j$ (considering the derivative graph of Γ we see that this can be done in time quadratic in $\text{size}(\Gamma)$). For each edge E_i , $1 \leq i \leq m$, denote its label by u_i , its origin by $V_{o(i)}$, and its terminus by $V_{t(i)}$. We similarly assume that $o(1) = 1$ and $t(m) = n$.

Next, we produce in polynomial time n freely independent elements v_1, \dots, v_n of the free group $F_2 = \langle x, y \rangle$, of which we think as labels of the corresponding vertices V_1, \dots, V_n . For example, $v_j = x^j y x^j$, $j = 1, \dots, n$, suffice. We claim that

$$g_1 = v_{o(1)} u_1 v_{t(1)}^{-1}, g_2 = v_{o(2)} u_2 v_{t(2)}^{-1}, \dots, g_m = v_{o(m)} u_m v_{t(m)}^{-1}; g = v_1 v_n^{-1}$$

is a positive instance of **SSP**($G * F_2$) (or **SSP**($G \times F_2$)) if and only if Γ is positive instance of **AGWP**(G). (See Figure 4 for an example.) Indeed, suppose there is an

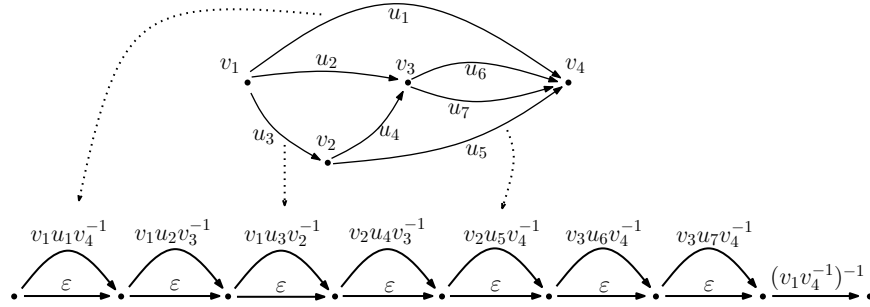


FIGURE 4. Reduction of **AGWP**(G) to **SSP**($G * F_2$) and **SSP**($G \times F_2$). Dotted arrows illustrate the correspondence between edges labeled by u_i and elements $g_i = v_{o(i)} u_i v_{t(i)}^{-1}$.

edge path E_{i_1}, \dots, E_{i_k} from $\alpha = V_1$ to $\omega = V_n$ in Γ with $u_{i_1} \cdots u_{i_k} = 1$ in G . Note that since the above sequence of edges is a path, for each $1 \leq \mu \leq k-1$, we have $V_{t(\mu)} = V_{o(\mu+1)}$, so $v_{t(\mu)} = v_{o(\mu+1)}$. Then the same choice of elements g_{i_1}, \dots, g_{i_k} gives

$$\begin{aligned} g_{i_1} \cdots g_{i_k} &= (v_{o(i_1)} u_{i_1} v_{t(i_1)}^{-1}) (v_{o(i_2)} u_{i_2} v_{t(i_2)}^{-1}) \cdots (v_{o(i_k)} u_{i_k} v_{t(i_k)}^{-1}) \\ &= v_{o(i_1)} u_{i_1} (v_{t(i_1)}^{-1} v_{o(i_2)}) u_{i_2} (v_{t(i_2)}^{-1} v_{o(i_3)}) \cdots v_{o(i_k)} u_{i_k} v_{t(i_k)}^{-1} \\ &= v_{o(i_1)} u_{i_1} u_{i_2} \cdots u_{i_k} v_{t(i_k)}^{-1} \\ &= v_{o(i_1)} v_{t(i_k)}^{-1} = v_1 v_n^{-1} = g. \end{aligned}$$

In the opposite direction, suppose in $G * F_2$ (or $G \times F_2$) the equality

$$(3) \quad g_{i_1} \cdots g_{i_k} = g, \quad i_1 < \dots < i_k,$$

takes place. Consider the F_2 -component of this equality:

$$v_{o(i_1)} v_{t(i_1)}^{-1} \cdot v_{o(i_2)} v_{t(i_2)}^{-1} \cdots v_{o(i_k)} v_{t(i_k)}^{-1} = v_1 v_n^{-1}.$$

Since v_1, \dots, v_n are freely independent, it is easy to see by induction on k that the latter equality is only possible if

$$v_{o(i_1)} = v_1, v_{t(i_1)} = v_{o(i_2)}, v_{t(i_2)} = v_{o(i_3)}, \dots, v_{t(i_{k-1})} = v_{o(i_k)}, v_{t(i_k)} = v_n,$$

i.e. edges $E_{i_1}, E_{i_2}, \dots, E_{i_k}$ form a path from V_1 to V_n in Γ . Further, inspecting the G -component of the equality (3), we get that $u_{i_1} u_{i_2} \dots u_{i_k} = 1$, as required in **AGWP**(G). \square

3. **AGWP** AND **SSP** IN DIRECT PRODUCTS.

We show in this section that direct product of groups may change the complexity of the subset sum problem (**SSP**) and the acyclic graph word problem (**AGWP**) dramatically, in contrast with results of Section 4, where we show that free products preserve the complexity of **AGWP**.

Proposition 6. There exist groups G, H such that **AGWP**(G), **AGWP**(H) $\in \mathbf{P}$, but **AGWP**($G \times H$) is **NP**-complete.

Proof. It was shown in [14, Theorem 7.4] that **BSMP**($F_2 \times F_2$) is **NP**-complete. By Proposition 4 it follows that **AGWP**($F_2 \times F_2$) is **NP**-complete, while by Proposition 3 **AGWP**(F_2) $\in \mathbf{P}$. \square

A similar statement can be made about **SSP** with the help of Proposition 5(2). However, in the next proposition we organize reduction of **BSMP**(G) to **SSP**($G \times \mathbb{Z}$), thus simplifying the “augmenting” group, which allows to make a slightly stronger statement about complexity of **SSP** in direct products. We remind that the definition of **P**-time Cook reduction used in the statement below can be found in Section 1.1.

Proposition 7. Let G be a finitely generated monoid. Then **BSMP**(G) **P**-time Cook reduces to **SSP**($G \times \mathbb{Z}$).

Proof. The proof below is for the case of a group G . The case of a monoid G is treated in the same way with obvious adjustments.

Let $w_1, w_2, \dots, w_k, w, 1^n$ be the input of **BSMP**(G). We construct graphs Γ_m , $m = 1, \dots, n$, with edges labeled by elements of $G \times \mathbb{Z}$ as shown in the Figure 5. Note that a path from α to ω is labeled by a word trivial in $G \times \mathbb{Z}$ if and only if it passes through exactly m edges labeled by $(w_{i_1}, 1), \dots, (w_{i_m}, 1)$ and $w_{i_1} \dots w_{i_m} = w$ in G . Therefore, the tuple $w_1, \dots, w_k, 1^n$ is a positive instance of **BSMP**(G) if and only if at least one of graphs $\Gamma_1, \dots, \Gamma_n$ is a positive instance of **SSP**($G \times \mathbb{Z}$). \square

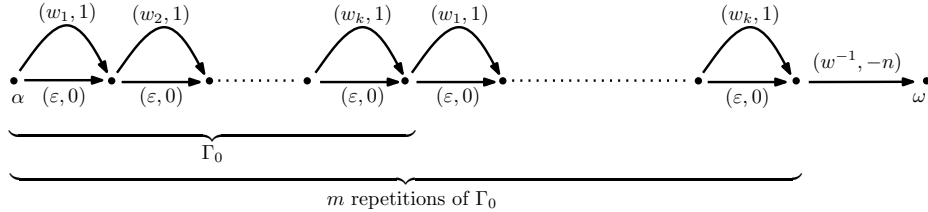


FIGURE 5. Reduction of **BSMP**(G) to **SSP**($G \times \mathbb{Z}$). Graph Γ_m .

We put $G = F_2 \times F_2$ in the above Proposition 7 to obtain the following result.

Proposition 8. $\mathbf{SSP}(F_2 \times F_2 \times \mathbb{Z})$ is **NP**-complete.

Proof. As we mentioned in the proof of Proposition 6, $\mathbf{AGWP}(F_2 \times F_2)$ is **NP**-complete. By Proposition 7, the latter **P**-time Cook reduces to $\mathbf{SSP}((F_2 \times F_2) \times \mathbb{Z})$. Therefore, the $\mathbf{SSP}(F_2 \times F_2 \times \mathbb{Z})$ is **NP**-complete. \square

The latter proposition answers the question whether direct product preserves polynomial time **SSP**.

Corollary 3.1. There exist finitely generated groups G, H such that $\mathbf{SSP}(G) \in \mathbf{P}$, $\mathbf{SSP}(H) \in \mathbf{P}$ but $\mathbf{SSP}(G \times H)$ is **NP**-complete.

Proof. By Corollary 2.2, $\mathbf{AGWP}(F_2 \times \mathbb{Z})$ is in **P**. Therefore, $\mathbf{SSP}(F_2)$ and $\mathbf{SSP}(F_2 \times \mathbb{Z})$ are in **P**, while $\mathbf{SSP}(F_2 \times (F_2 \times \mathbb{Z}))$ is **NP**-complete by the above result. \square

Corollary 3.2. **SSP** is **NP**-complete in braid groups B_n , $n \geq 7$, special linear groups $SL(n, \mathbb{Z})$, $n \geq 5$, graph groups whose graph contains the square pyramid \boxtimes (also called the wheel graph W_5) as an induced subgraph.

Proof. Note that $F_2 \times F_2$ embeds in a braid group B_n with $n \geq 5$ by a result of Makanina [9]. The statement now follows from Propositions 7 and 1 since all of the listed groups contain $F_2 \times F_2 \times \mathbb{Z}$ as a subgroup. \square

It remains to be seen whether $\mathbf{SSP}(F_2 \times F_2)$ is **NP**-complete.

4. **AGWP** AND RATIONAL SUBSET MEMBERSHIP PROBLEM IN FREE PRODUCTS WITH FINITE AMALGAMATED SUBGROUPS

Our primary concern in this section is the complexity of acyclic graph word problem (**AGWP**) in free products of groups. However, our approach easily generalizes to free products with amalgamation over a finite subgroup, and to a wider class of problems. We remind that the definition of **P**-time Cook reduction can be found in Section 1.1.

Theorem 4.1. Let G, H be finitely generated groups, and C be a finite group that embeds in G, H . Then $\mathbf{AGWP}(G *_C H)$ is **P**-time Cook reducible to $\mathbf{AGWP}(G), \mathbf{AGWP}(H)$.

Proof. Let G be given by a generating set X , and H by Y . Let $\Gamma = \Gamma_0$ be the given acyclic graph labeled by $\Sigma = X \cup X^{-1} \cup Y \cup Y^{-1} \cup C$ (we assume that alphabets $X^{\pm 1}, Y^{\pm 1}$ are disjoint from C). Given the graph Γ_k , $k \in \mathbb{Z}$, construct graph Γ_{k+1} by adding edges to Γ_k as follows.

Consider Γ'_k , the maximal subgraph of Γ_k labeled by $X \cup X^{-1} \cup C$ (i.e., the graph obtained by removing all edges labeled by $Y \cup Y^{-1}$). For each $c \in C$ and each pair of vertices $v_1, v_2 \in V(\Gamma_k)$ from the same connected component of Γ_k , decide whether a word equal to c in G is readable as a label of an oriented path in Γ'_k , using the solution to $\mathbf{AGWP}(G)$. For $c \in C$, let E_c be the set of pairs $(v_1, v_2) \in V(\Gamma_k) \times V(\Gamma_k)$ such that the answer to the above question is positive, and there is no edge $v_1 \xrightarrow{c} v_2$ in Γ_k . Construct the graph $\bar{\Gamma}_k$ by adding edges $v_1 \xrightarrow{c} v_2$, $(v_1, v_2) \in E_c$ to the graph Γ_k , for all $c \in C$. Now consider the maximal subgraph Γ''_k of $\bar{\Gamma}_k$ labeled by $Y \cup Y^{-1} \cup C$ and perform similar operation using the solution to $\mathbf{AGWP}(H)$, obtaining the graph $\bar{\bar{\Gamma}}_k = \Gamma_{k+1}$. Since there are at most $2|C| \cdot |V(\Gamma)|^2$ possible c -edges to be drawn, it follows that $\text{size}(\Gamma_{k+1}) < 2|C| \text{size}(\Gamma)^2$, and that $\Gamma_k = \Gamma_{k+1} = \dots$ for some $k = n$, where $n \leq 2|C| \text{size}(\Gamma)^2$.

We claim that a word w equal to 1 in $G *_C H$ is readable from α to ω in Γ if and only if there is an edge $\alpha \xrightarrow{\varepsilon} \omega$ in the graph Γ_n . Indeed, suppose there is a path in Γ and, therefore, in Γ_n , from α to ω labeled by a word $w = w_1 w_2 \cdots w_m$, with $w_j \in \Sigma$ and at least one non- C letter among w_1, \dots, w_m , such that $w = 1$ in $G *_C H$. The normal form theorem for free products with amalgamated subgroup guarantees that w has a subword $w' = w_i w_{i+1} \cdots w_j$ of letters in $X \cup X^{-1} \cup C$ or $Y \cup Y^{-1} \cup C$ with $w' = c \in C$ in G or H , respectively, with at least one non- C letter among w_i, \dots, w_j . Since $\Gamma_n = \Gamma_{n+1}$, the word $w_1 \cdots w_{i-1} c w_{j+1} \cdots w_m$ is readable as a label of a path in Γ_n from α to ω . By induction, a word $c_1 \cdots c_\ell$, $\ell \leq m$, $c_1, \dots, c_\ell \in C$, is readable as a label of an oriented path in Γ_n from α to ω . By the construction, $\Gamma_{n+1} = \Gamma_n$ contains an edge $\alpha \xrightarrow{\varepsilon} \omega$. The converse direction of the claim is evident. \square

Corollary 4.2. Let G, H be finitely generated groups, and C be a finite group that embeds in G, H . If $\mathbf{AGWP}(G), \mathbf{AGWP}(H) \in \mathbf{P}$ then $\mathbf{AGWP}(G *_C H) \in \mathbf{P}$.

Corollary 4.3. Subset sum (**SSP**), bounded knapsack (**BKP**), bounded submonoid membership (**BSMP**), and acyclic graph word (**AGWP**) problems are polynomial time decidable in any finite free product with finite amalgamations of finitely generated virtually nilpotent and hyperbolic groups.

Theorem 4.1 can be generalized to apply in the following setting. We say that a family \mathcal{F} of finite directed graphs is *progressive* if it is closed under the following operations:

- (1) taking subgraphs,
- (2) adding shortcuts (i.e., drawing an edge from the origin of an oriented path to its terminus), and
- (3) appending a hanging oriented path by its origin.

Two proper examples of such families are acyclic graphs, and graphs stratifiable in a sense that vertices can be split into subsets V_1, \dots, V_n so that edges only lead from V_k to $V_{\geq k}$. We say that a subset of a group G is \mathcal{F} -rational if it can be given by a finite (non-deterministic) automaton in \mathcal{F} . Finally, by \mathcal{F} -uniform rational subset membership problem for G we mean the problem of establishing, given an \mathcal{F} -rational subset of a group G and a word as an input, whether the subset contains an element equal to the given word in G . In this terminology, $\mathbf{AGWP}(G)$ is precisely the $\mathcal{F}_{\text{acyc}}$ -uniform rational subset membership problem for G , where $\mathcal{F}_{\text{acyc}}$ is the family of acyclic graphs.

Theorem 4.4. Let \mathcal{F} be a progressive family of directed graphs. Let G, H be finitely generated groups, and C be a finite group that embeds in G, H . Then \mathcal{F} -uniform rational subset membership problem for $G *_C H$ is \mathbf{P} -time Cook reducible to that for G and H .

Proof. Given an automaton Γ and a word w as an input, we start by forming an automaton Γ_w by appending a hanging path labeled by w^{-1} at every accepting state of Γ . The procedure then repeats that in the proof of Theorem 4.1, with obvious minor adjustments. \square

In particular, uniform rational subset membership problem for $G *_C H$, with C finite, is \mathbf{P} -time Cook reducible to that in the factors G and H , cf. decidability results in the case of graphs of groups with finite edge groups in [5].

Given that the uniform rational subset membership problem is \mathbf{P} -time decidable for free abelian groups [6], and that \mathbf{P} -time decidability of the same problem carries from a finite index subgroup [2], the above theorem gives the following corollary.

Corollary 4.5. Let G be a finite free product with finite amalgamations of finitely generated abelian groups. The rational subset membership problem in G is decidable in polynomial time.

5. KNAPSACK PROBLEM IN FREE PRODUCTS

As examples cited in Section 1.1 show, the (unbounded) knapsack problem (**KP**) in general does not reduce to its bounded version (**BKP**), nor to acyclic graph word problem (**AGWP**). However, it was shown in [14] that in the case of hyperbolic groups there is indeed such reduction. In light of results of Section 4, it is natural to ask whether a similar reduction takes place for free products of groups.

For an element f of a free product of groups $G * H$, let $\|f\|$ denote the *syllable length* of f , i.e. its geodesic length in generators $G \cup H$ of $G * H$. We say that f is *simple* if f can be conjugated by an element of $G * H$ into $G \cup H$, i.e.

$$(4) \quad f = u^{-1} f' u,$$

where $u \in G * H$, $\|f'\| \leq 1$. Otherwise, we say that f is *non-simple*.

Lemma 5.1. Let G, H be groups and let an element $f \in G * H$ be non-simple. Then there are $a_1, \dots, a_r \in G \cup H$, $b_1, \dots, b_s \in G \cup H$, $c_1, \dots, c_t \in G \cup H$, $r + t \leq 2\|f\|$, $s \leq \|f\|$, such that for every integer $n \geq 3$ the normal form of f^n is

$$a_1 \cdots a_r (b_1 \cdots b_s)^{n-2} c_1 \cdots c_t.$$

Proof. Follows from the normal form theorem for free products. \square

The statement below is, in some sense, a strengthened big-powers condition for a free product.

Proposition 9. Let $p(x)$ be the polynomial $p(x) = x^2 + 4x$. Let G, H be groups. If for $f_1, f_2, \dots, f_m, f \in G * H$ there exist non-negative integers $n_1, n_2, \dots, n_m \in \mathbb{Z}$ such that

$$(5) \quad f_1^{n_1} f_2^{n_2} \cdots f_m^{n_m} = f$$

then there exist such integers n_1, \dots, n_m with

$$n_i \leq p(\|f_1\| + \|f_2\| + \dots + \|f_m\| + \|f\|) \text{ whenever } f_i \text{ is non-simple.}$$

Proof. For convenience, denote $\|f_1\| + \|f_2\| + \dots + \|f_m\| + \|f\| = N$ and $f = f_0$, $-1 = n_0$. For each f_i , $i = 0, \dots, m$, we represent $f_i^{n_i}$ by its normal form w_i , using Lemma 5.1 whenever f_i is non-simple and $n_i \geq 3$. Suppose that for some non-simple f_i , $n_i > N^2 + 4N$. We inspect the path traversed by the word $w_1 w_2 \dots w_m w_0$ in the Cayley graph of $G * H$ with respect to generators $G \cup H$. Since this word corresponds to the trivial group element, the path must be a loop and thus the word w_i splits in at most m pieces, each piece mutually canceling with a subword of w_j , $j \neq i$. Let $w_i = ab^{n_i-2}c$ as in Lemma 5.1. Then by pigeonhole principle at least one piece contains at least $(n_i - 2 - m)/m \geq N + 1$ copies of the word b . Observe that f_j is non-simple and $n_j \geq 3$, otherwise $\|f_j^{n_j}\| \leq 2N$, so it cannot mutually cancel with b^{N+1} since $\|b^{N+1}\| \geq 2N + 2$. Let $w_j = a'b'^{n_j-2}c'$. We note that $\|b\| \leq \|f_i\| \leq N$, $\|b'\| \leq \|f_j\| \leq N$, so $\|b'\| \leq N$ copies of b mutually cancel

with $\|b'\| \cdot \|b\|/\|b'\| = \|b\|$ copies of b' (up to a cyclic shift). Therefore replacing n_i, n_j with $n_i - \|b'\|$ and $n_j + \|b\|$, respectively, preserves equality (5). Iterating this process, we may find numbers n_1, \dots, n_m that deliver equality (5) such that $n_i \leq p(N) = N^2 + 4N$ whenever f_i is non-simple. \square

We observe that $p(N) = N^2 + 4N$ in the above proposition is monotone on $\mathbb{Z}_{\geq 0}$.

We say that G is a *polynomially bounded knapsack group* if there is a polynomial q such that any instance (g_1, \dots, g_k, g) of $\mathbf{KP}(G)$ is positive if and only if the instance $(g_1, \dots, g_k, g, q(N))$ of $\mathbf{BKP}(G)$ is positive, where N is the total length of g_1, \dots, g_k, g . It is easy to see that this notion is independent of a choice of a finite generating set for G .

Proposition 10. If G, H are polynomially bounded knapsack groups then $G * H$ is a polynomially bounded knapsack group.

Proof. Let $f_1, f_2, \dots, f_m, f \in G * H$ be an input of $\mathbf{KP}(G * H)$. For each f_i , normal form of $f_i^{n_i}$, $n_i \geq 3$, is given by Lemma 5.1 or by simplicity of f_i . Suppose some n_1, n_2, \dots, n_m provide a solution to \mathbf{KP} , i.e. $f_1^{n_1} \dots f_m^{n_m} f^{-1} = 1$ in $G * H$. Representing f^{-1} and each $f_i^{n_i}$ by their normal forms and combining like terms we obtain (without loss of generality) a product

$$(6) \quad g_1 h_1 g_2 h_2 \dots g_\ell h_\ell = 1,$$

where each $g_i \in G$ and each $h_i \in H$, and we may assume $\ell \leq \|f\| + m \cdot (\max\{\|f_i\|\}) \cdot p(\max\{\|f_i\|\})$ by Proposition 9, so $\ell \leq N^2 p(N)$ where N is the total length of the input (i.e. the sum of word lengths of f_1, \dots, f_m, f). Further, since the product in (6) represents the trivial element, it can be reduced to 1 by a series of eliminations of trivial syllables and combining like terms:

$$\begin{aligned} g_1 h_1 g_2 h_2 \dots g_\ell h_\ell &= g_1^{(1)} h_1^{(1)} \dots g_\ell^{(1)} h_\ell^{(1)} \\ &= g_1^{(2)} h_1^{(2)} \dots g_{\ell-1}^{(2)} h_{\ell-1}^{(2)} \\ &= \dots = g_1^{(\ell)} h_1^{(\ell)} = 1, \end{aligned}$$

where the product labeled by (j) is obtained from the one labeled by $(j-1)$ by a single elimination of a trivial term and combining the two (cyclically) neighboring terms. Observe that each $g_i^{(j)}$ is (up to a cyclic shift) a product of the form $g_\alpha g_{\alpha+1} \dots g_\beta$; similarly for $h_i^{(j)}$. Therefore, each $g_i^{(j)}$ is of the form

$$(7) \quad g_i^{(j)} = d_{1j}^{\delta_{1j}} d_{2j}^{\delta_{2j}} \dots d_{k_{ij},j}^{\delta_{k_{ij},j}},$$

where each $d_{\mu j}$, $1 \leq \mu \leq k_{ij}$ is either

- (NS) one of the syllables involved in Lemma 5.1, or normal form of some f_i or f_i^2 , or “u” part of (4), in which case $\delta_{\mu j} = 1$, or
- (S) the syllable f' in (4) for some f_ν , in which case $\delta_{\mu j} = n_\nu$.

On the one hand, the total amount of syllables $d_{\mu j}$ involved in (7) for a fixed j is $k_{1j} + k_{2j} + \dots + k_{\ell-j+1,j}$. On the other hand, it cannot exceed $k_{11} + k_{21} + \dots + k_{\ell 1}$ since $g_1^{(j)}, g_2^{(j)}, \dots, g_{\ell-j+1}^{(j)}$ are obtained by eliminating and combining elements g_1, g_2, \dots, g_ℓ . Taking into account that each $k_{i1} \leq m + 1 \leq N$, we obtain $k_{1j} + k_{2j} + \dots + k_{\ell-j+1,j} \leq \ell N \leq N^3 p(N)$.

Now, given the equality $g_i^{(j)} = 1$ for some i, j , if the option (S) takes place for any $1 \leq \mu \leq k_{ij}$, then the right hand side of the corresponding equality (7) can be represented (via a standard conjugation procedure) as a positive instance of the $\mathbf{KP}(G)$ with input of length bounded by $(Nk_{ij})^2 \leq N^8 p^2(N)$. Since G is a polynomially bounded knapsack group, we may assume that every n_ν that occurs as some $\delta_{\mu j}$ in some $g_i^{(j)}$ is bounded by a polynomial $p_G(N)$. Similar argument takes place for the H -syllables $h_i^{(j)}$, resulting in a polynomial bound $p_H(N)$.

It is only left to note that since for every $1 \leq \nu \leq m$, either f_ν is non-simple and then n_ν is bounded by $p(N)$, or it is simple and then n_ν is bounded by $p_G(N)$ or $p_H(N)$, so every n_ν is bounded by $p(N) + p_G(N) + p_H(N)$. \square

Theorem 5.2. *If G, H are polynomially bounded knapsack groups such that $\mathbf{AGWP}(G), \mathbf{AGWP}(H) \in \mathbf{P}$ then $\mathbf{KP}(G * H) \in \mathbf{P}$.*

Proof. By Proposition 10 $\mathbf{KP}(G * H)$ is \mathbf{P} -time reducible to $\mathbf{BKP}(G * H)$. In turn, the latter is \mathbf{P} -time reducible to $\mathbf{AGWP}(G * H)$ by Proposition 4. Finally, $\mathbf{AGWP}(G * H) \in \mathbf{P}$ by Corollary 4.2. \square

Corollary 5.3. \mathbf{KP} is polynomial time decidable in free products of finitely generated abelian and hyperbolic groups in any finite number.

Proof. By [14, Theorem 6.7], hyperbolic groups are polynomially bounded knapsack groups. By [1], so are finitely generated abelian groups. The statement follows by Theorem 5.2. \square

The authors are grateful to A. Myasnikov for bringing their attention to the problem and for insightful advice and discussions, and to the anonymous referees for their astute observations and helpful suggestions.

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